Journal of Engineering Mathematics, Vol. 16, No. 1, January 1982

The Lagrangian formalism and the balance law of angular momentum for a rigid body

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(Received April 21, 1981)

SUMMARY

In this paper we derive Lagrange's equations for a rigid body rotating about its centre of gravity. We base our calculations on the balance law of angular momentum and refrain from the use of the customary and deficient model in which a rigid body is regarded as a limiting case of a rigid system of mass points. After some kinematical preliminaries we proceed in two ways. First we apply coordinates to achieve our goal, and in a second approach we do without them. The latter analysis is seen to be very simple.

1. Introduction

The literature abounds in textbooks on classical mechanics, treating the balance laws of mass points and rigid bodies. As a rule Lagrange's equations are presented for systems with constraints. In general the methods used are standard and well-established, and usually much attention is paid to applications in science and engineering. We note that recently some books have appeared the authors of which reformulate the foundations of classical mechanics using contemporary concepts from the calculus of manifolds. In this relation we mention the monograph of Arnold [1], which excels in rigour and succintness.

However, there is one property, in our view a conceptual shortcoming, these books have in common. In deriving the two balance laws of a rigid body, viz., the balance of linear momentum and the balance of angular momentum (with respect to a fixed point of reference or to the centre of gravity of the body), only the balance law of linear momentum of a mass point is invoked. As is well-known this is achieved by assuming a rigid body to consist of (an infinite number of) particles exerting central forces on each other in pairs. We are unable to understand this model. It would appear that it bears upon a medium provided with a microstructure in the form of a system of entangled spatial trusses: mass points connected by an infinite number of massless springs, the stiffness of which is to be increased beyond all limits. Since this limiting process is incompatible with the absence of forces other than central ones, in our view the model is deficient.

We meet the same difficulty when we try to follow the usual derivation of Lagrange's equations for a system with constraints from the balance law valid for a mass point only. The seeming simplicity of the ensuing analysis and, as we presume, the influence of Whittaker's famous treatise [2] have induced that the dubious model found general favour here as well. However, there is a number of authors which work the other way round. After having formulated the balance laws for points and bodies, they present Lagrange's equations more or less independently of the foregoing text. Checking these equations in the case of mass points and rigid bodies they treat the internal forces in the latter as reaction forces, which by definition do not contribute to virtual work. This procedure, which works only one-way, can be accepted.

Yet we have found a rational analysis leading from the balance laws to Lagrange's equations and vice versa only in a recent book written by Wang [3]. He accepts rigid bodies as primitive concepts like mass points and assigns one balance law, viz., that of linear momentum to a mass point and two balance laws, viz., that of linear momentum and that of angular momentum, to a rigid body. Considering a general system consisting of particles and rigid bodies with constraints, he subsequently establishes the exact and complete interrelations between the balance laws and the Lagrangian formalism. However, we note that his analysis is more involved than the traditional one. We grant that for the greater part this is due to the fact that the kinematical description of the rotation of a rigid body demands a more elaborate formalism than that of its translation.

In view of this we have tried to find an alternative analysis, simpler and clear, with the ultimate aim to oppose further application of the vague and improper model of a rigid body. In this paper we reconsider the problem of deriving Lagrange's equations starting from the balance laws as Wang has formulated them in [3]. Then, in order to avoid a cumbersome notation, we have singled out the rotational motion of one body about its centre of gravity. (The analysis for mass points and for the translation of the centre of gravity of a body can be accomplished in the customary way, as will be touched upon in Section 5). After a short description of the kinematics of a rigid body in Section 2 and the introduction of virtual displacements and kinetic energy in Sections 3 and 4, respectively, we define our problem in Section 5. Then we proceed along two different lines. As Wang [3] applies Eulerian angles as coordinates for the rotational motion, in Section 6 we make use of the orthogonal matrix associated with the rotation, the elements of which are referred to a fixed Cartesian system of axes without the intervention of Eulerian angles. In an alternative approach (Section 7) we refrain from the use of any coordinate system, (obviously with the exception of the concept of generalized coordinates), and apply some simple identities from vector analysis.

2. Kinematics of a rigid body

In this section we consider a rigid body $\mathscr{B}$ moving in a three-dimensional space. We refer the motion to a rectangular Cartesian coordinate system x_1, x_2, x_3, which is fixed in space and has origin O and base vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, (Fig. 2.1). A second rectangular Cartesian coordinate system ξ_1, ξ_2, ξ_3, is introduced that moves with $\mathscr{B}$ and has base vectors $\mathbf{f}_1$, $\mathbf{f}_2$, $\mathbf{f}_3$, and origin o. The relation between the x_i- and the ξ_i- coordinates of any point P of $\mathscr{B}$ is

$$x_i = c_i + Q_{ij}\xi_j, \quad i = 1, 2, 3, \tag{2.1}$$

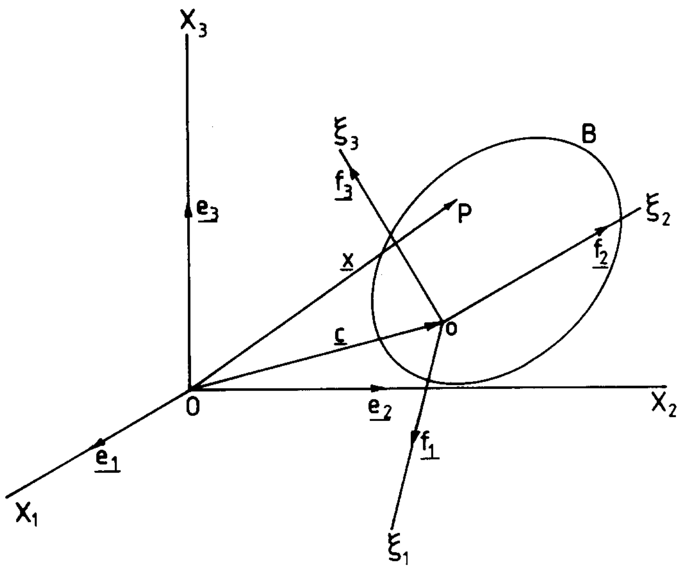

Figure 2.1. The coordinate system $OX_1X_2X_3$ is fixed in space; the coordinate system $o\xi_1\xi_2\xi_3$ moves with the body $\mathscr{B}$.

in which

x_i = coordinate with respect to the fixed basis,
ξ_i = coordinate with respect to the moving basis,
c_i = coordinate of o with respect to the fixed basis,
Q_{ij} = element of a proper orthogonal matrix Q, i.e.,

$$QQ^T = Q^TQ = I \quad \text{and} \quad \det Q = +1. \tag{2.2}$$

We call ξ_i the convective coordinates. Obviously they are constant during the motion. In (2.1) and subsequent expressions we apply the summation convention, implying summation over doubly repeated indices. Latin indices will assume the values 1, 2 and 3. The relation (2.1) can be written in vector form as follows

$$\mathbf{x} = \mathbf{c} + Q\boldsymbol{\xi}, \tag{2.3}$$

with $\mathbf{x} = x_i\mathbf{e}_i$, $\mathbf{c} = c_i\mathbf{e}_i$, $\boldsymbol{\xi} = \xi_i\mathbf{f}_i$ and $Q = Q_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$. The relation (2.3) can be inverted

$$\boldsymbol{\xi} = Q^T(\mathbf{x} - \mathbf{c}). \tag{2.4}$$

We describe a motion $\mathbf{x} = \tilde{\mathbf{x}}(t)$ of $\mathscr{B}$ by prescribing the following functions of the time $t \in \mathbb{R}$

$$\mathbf{c} = \tilde{\mathbf{c}}(t) \quad \text{and} \quad Q = \tilde{Q}(t), \tag{2.5}$$

so that with (2.3) we have

$$\tilde{\mathrm{x}}(t) = \tilde{\mathrm{c}}(t) + \tilde{Q}(t)\xi. \tag{2.6}$$

In view of $(2.2)^1$ we find

$$\dot{\tilde{Q}}\tilde{Q}^T + \tilde{Q}\dot{\tilde{Q}}^T = 0, \tag{2.7}$$

where we have written for short $\dot{\tilde{Q}} = d\tilde{Q}(t)/dt$ and have omitted the argument t. (When there is no risk of confusion we shall do this in subsequent expressions as well).

Evidently, the tensor

$$\tilde{\Omega} = \dot{\tilde{Q}}\tilde{Q}^T \tag{2.8}$$

is skew-symmetric. We call $\Omega = \tilde{\Omega}(t)$ the spin tensor. Its axial vector $\omega = \tilde{\omega}(t)$, satisfying

$$\forall_{\mathbf{a}} \quad \omega \wedge \mathrm{a} = \Omega\mathbf{a}, \tag{2.9}$$

is the angular velocity vector of the body. From (2.9) we find the following two relations between the components of Ω and ω with respect to any rectangular Cartesian basis

$$\Omega_{ij} = -e_{ijk}\,\omega_k \quad \text{and} \quad \omega_i = -\tfrac{1}{2}e_{ijk}\,\Omega_{jk}, \tag{2.10}$$

where e_{ijk} denotes the permutation symbol. For the later use we mention the relation

$$\dot{\tilde{Q}} = -\tilde{Q}\dot{\tilde{Q}}^T\tilde{Q}, \tag{2.11}$$

which is a consequence of (2.7).

Differentiating (2.6) with respect to t and using (2.4), (2.8) and (2.9) yields the following expression for the velocity v

$$\mathrm{v} = \tilde{\mathrm{v}}(t) = \dot{\tilde{\mathrm{x}}}(t) = \dot{\tilde{\mathrm{c}}}(t) + \tilde{\omega}(t) \;\wedge\; \{\tilde{\mathrm{x}}(t) - \tilde{\mathrm{c}}(t)\}. \tag{2.12}$$

At this stage we introduce the concept of generalized coordinates $q^\alpha \in \mathbb{R}$, $\alpha = 1, 2, ..., n$. We denote the n-dimensional vector $(q^1, q^2, ..., q^n)$ by $\mathrm{q} \in \mathbb{R}^n$. In the usual way we assume that the instantaneous position of the body $\mathscr{B}$ is determined by the values assigned to these n parameters and the time t. This means that the following functions exist

$$\mathrm{c} = \mathrm{c}(\mathrm{q}, t) \quad \text{and} \quad Q = Q(\mathrm{q}, t). \tag{2.13}$$

If t occurs as a parameter explicitly, the system is called rheonomic; if not, then it is scleronomic. To define a motion (2.5) we have to prescribe the functions

$$q^\alpha = \tilde{q}^\alpha(t), \quad \alpha = 1, 2, ..., n. \tag{2.14}$$

We note the identities

$$\tilde{c}(t) \equiv c(\tilde{q}(t), t), \tilde{Q}(t) \equiv Q(\tilde{q}(t), t) \quad \text{and} \quad \tilde{x}(t) \equiv x(\tilde{q}(t), t), \tag{2.15}$$

where the function $x(q, t)$ follows from (2.3) and (2.13),

$$x(q, t) = c(q, t) + Q(q, t)\,\xi. \tag{2.16}$$

In order to avoid a cumbersome notation we shall often omit the independent arguments in the sequel. For instance, if Q is considered as a function of t, then we write $\tilde{Q}$ instead of $\tilde{Q}(t)$. When Q is defined as in (2.13)[2], then we write Q (without upper tilde). The partial derivative $\partial Q(q, t)/\partial q^\alpha$ is abbreviated by $\partial Q/\partial q^\alpha$, etc. Finally we write $\frac{d}{dt}\left(\ldots\Big|_{q=\tilde{q}(t)}\right)$ in order to indicate that the differentiation with respect to t has to be performed after the substitution of $q^\alpha = \tilde{q}^\alpha(t)$ into the form between brackets.

Since Q is orthogonal we have analogous to (2.7)

$$\frac{\partial Q}{\partial q^\alpha} Q^T + Q \frac{\partial Q^T}{\partial q^\alpha} = 0, \quad \frac{\partial Q}{\partial t} Q^T + Q \frac{\partial Q^T}{\partial t} = 0. \tag{2.17}$$

For future use we derive from (2.17)[1]

$$\frac{\partial Q}{\partial q^\alpha} = -Q \frac{\partial Q^T}{\partial q^\alpha} Q. \tag{2.18}$$

Using (2.14) in (2.13), (2.12)[1], (2.8) and (2.10)[2] we find for the velocity

$$\tilde{v} = \dot{\tilde{x}} = \frac{\partial x}{\partial q^\alpha}\Big|_{q=\tilde{q}(t)} \dot{\tilde{q}}^\alpha + \frac{\partial x}{\partial t}\Big|_{q=\tilde{q}(t)}, \tag{2.19}$$

for the spin tensor

$$\tilde{\Omega} = \left(\frac{\partial Q}{\partial q^\alpha}\Big|_{q=\tilde{q}(t)} \dot{\tilde{q}}^\alpha + \frac{\partial Q}{\partial t}\Big|_{q=\tilde{q}(t)}\right) \tilde{Q}^T, \tag{2.20}$$

and for the angular velocity

$$\tilde{\omega}_i = -\tfrac{1}{2} e_{ijk} \left(\frac{\partial Q_{jl}}{\partial q^\alpha}\Big|_{q=\tilde{q}(t)} \dot{\tilde{q}}^\alpha + \frac{\partial Q_{jl}}{\partial t}\Big|_{q=\tilde{q}(t)}\right) \tilde{Q}_{kl}. \tag{2.21}$$

In these expressions and following ones summation over Greek indices runs from 1 to n. In view of the structure of these expressions, we define the following functions on the product space

$\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, the elements of which are the n generalized coordinates $q^\alpha \in \mathbb{R}$, $(\alpha = 1, 2, ..., n)$ the n generalized velocities $\dot{q}^\alpha \in \mathbb{R}$, $(\alpha = 1, 2, ..., n)$ and the time $t \in \mathbb{R}$:

the velocity
$$\mathrm{v}(\mathrm{q}, \dot{\mathrm{q}}, t) = \frac{\partial \mathrm{x}}{\partial q^\alpha} \dot{q}^\alpha + \frac{\partial \mathrm{x}}{\partial t}, \tag{2.22}$$

the spin tensor
$$\Omega(\mathrm{q}, \dot{\mathrm{q}}, t) = \left(\frac{\partial Q}{\partial q^\alpha} \dot{q}^\alpha + \frac{\partial Q}{\partial t}\right) Q^T, \tag{2.23}$$

and the angular velocity
$$\omega_i(\mathrm{q}, \dot{\mathrm{q}}, t) = -\tfrac{1}{2} e_{ijk} \left(\frac{\partial Q_{jl}}{\partial q^\alpha} \dot{q}^\alpha + \frac{\partial Q_{jl}}{\partial t}\right) Q_{kl}. \tag{2.24}$$

We note that in this context the quantities $\dot{q}^\alpha$ are completely arbitrary and that until further notice there is no relation between $\dot{q}^\alpha$ and $\dot{\tilde{q}}^\alpha(t)$. (Tentatively the dot in $\dot{q}^\alpha$ only has a symbolic meaning and the notation is of some use at a later stage when the value of $\dot{q}^\alpha$ is equated to $\dot{\tilde{q}}^\alpha(t)$ actually).

The following identities can be verified easily

$$\tilde{\mathrm{v}}(t) = \mathrm{v}(\tilde{\mathrm{q}}(t), \dot{\tilde{\mathrm{q}}}(t), t), \quad \tilde{\Omega}(t) = \Omega(\tilde{\mathrm{q}}(t), \dot{\tilde{\mathrm{q}}}(t), t), \quad \tilde{\omega}(t) = \omega(\tilde{\mathrm{q}}(t), \dot{\tilde{\mathrm{q}}}(t), t), \tag{2.25}$$

and

$$\frac{\partial \mathrm{v}}{\partial \dot{q}^\alpha} = \frac{\partial \mathrm{x}}{\partial q^\alpha}, \quad \frac{\partial \Omega}{\partial \dot{q}^\alpha} = \frac{\partial Q}{\partial q^\alpha} Q^T,$$
$$\frac{\partial \omega_i}{\partial \dot{q}^\alpha} = -\tfrac{1}{2} e_{ijk} \frac{\partial Q_{jl}}{\partial q^\alpha} Q_{kl}, \tag{2.26}$$
$$\left.\frac{\partial \mathrm{v}}{\partial q^\alpha}\right|_{\mathrm{q}=\tilde{\mathrm{q}}(t)} = \frac{d}{dt}\left(\left.\frac{\partial \mathrm{x}}{\partial q^\alpha}\right|_{\mathrm{q}=\tilde{\mathrm{q}}(t)}\right), \quad \left.\frac{\partial \mathrm{v}}{\partial t}\right|_{\mathrm{q}=\tilde{\mathrm{q}}(t)} = \frac{d}{dt}\left(\left.\frac{\partial \mathrm{x}}{\partial t}\right|_{\mathrm{q}=\tilde{\mathrm{q}}(t)}\right)$$

where we have deleted the independent arguments q, $\dot{\mathrm{q}}$ and t.

From $(2.26)^1$ and $(2.26)^4$ it follows that

$$\left.\frac{\partial \mathrm{v}}{\partial q^\alpha}\right|_{\mathrm{q}=\tilde{\mathrm{q}}(t)} = \frac{d}{dt}\left(\left.\frac{\partial \mathrm{v}}{\partial \dot{q}^\alpha}\right|_{\mathrm{q}=\tilde{\mathrm{q}}(t)}\right). \tag{2.27}$$

We conclude this section by giving two identities which are crucial in the proofs given in the final two sections of this paper:

$$\frac{\partial \mathrm{x}}{\partial q^\alpha} = \frac{\partial \mathrm{c}}{\partial q^\alpha} + \frac{\partial \omega}{\partial \dot{q}^\alpha} \wedge (\mathrm{x}-\mathrm{c}), \tag{2.28}$$

and

$$\left.\frac{\partial \omega}{\partial q^\alpha}\right|_{q=\tilde{q}(t)} + \left(\tilde{\omega} \wedge \left.\frac{\partial \omega}{\partial \dot{q}^\alpha}\right|_{q=\tilde{q}(t)}\right) = \frac{d}{dt}\left(\left.\frac{\partial \omega}{\partial \dot{q}^\alpha}\right|_{q=\tilde{q}(t)}\right). \tag{2.29}$$

To prove (2.28) we substitute (2.16) into (2.22) and get through the use of (2.4) and (2.9)

$$\mathbf{v}(\mathbf{q}, \dot{\mathbf{q}}, t) = \frac{\partial \mathbf{c}}{\partial q^\alpha}\dot{q}^\alpha + \frac{\partial \mathbf{c}}{\partial t} + \omega \wedge (\mathbf{x}-\mathbf{c}). \tag{2.30}$$

This relation and (2.26)[1] lead to (2.28). In order to prove (2.29) we proceed as follows. The expression (2.30) is used in the left-hand side of (2.27) and subsequently (2.28) is applied, yielding

$$\frac{\partial \mathbf{v}}{\partial q^\alpha} = \frac{\partial^2 \mathbf{c}}{\partial q^\alpha \partial q^\beta}\dot{q}^\beta + \frac{\partial^2 \mathbf{c}}{\partial q^\alpha \partial t} + \frac{\partial \omega}{\partial q^\alpha} \wedge (\mathbf{x}-\mathbf{c}) + \omega \wedge \left(\frac{\partial \omega}{\partial \dot{q}^\alpha} \wedge (\mathbf{x}-\mathbf{c})\right). \tag{2.31}$$

Substituting (2.30) into the right-hand side of (2.27) and using (2.12) we get

$$\begin{aligned}\frac{d}{dt}\left(\left.\frac{\partial \mathbf{v}}{\partial \dot{q}^\alpha}\right|_{q=\tilde{q}(t)}\right) &= \frac{d}{dt}\left[\left.\frac{\partial \mathbf{c}}{\partial q^\alpha}\right|_{q=\tilde{q}(t)} + \left.\frac{\partial \omega}{\partial \dot{q}^\alpha}\right|_{q=\tilde{q}(t)} \wedge (\tilde{\mathbf{x}}-\tilde{\mathbf{c}})\right] \\ &= \left.\frac{\partial^2 \mathbf{c}}{\partial q^\alpha \partial q^\beta}\right|_{q=\tilde{q}(t)}\dot{\tilde{q}}^\beta + \left.\frac{\partial^2 \mathbf{c}}{\partial q^\alpha \partial t}\right|_{q=\tilde{q}(t)} + \frac{d}{dt}\left(\left.\frac{\partial \omega}{\partial \dot{q}^\alpha}\right|_{q=\tilde{q}(t)}\right) \wedge (\tilde{\mathbf{x}}-\tilde{\mathbf{c}}) \\ &\quad + \left.\frac{\partial \omega}{\partial \dot{q}^\alpha}\right|_{q=\tilde{q}(t)} \wedge (\tilde{\omega} \wedge (\tilde{\mathbf{x}}-\tilde{\mathbf{c}})).\end{aligned} \tag{2.32}$$

Then we substitute (2.14) into (2.31) and subtract (2.32). In view of (2.27) we arrive at

$$\left[\frac{d}{dt}\left(\left.\frac{\partial \omega}{\partial \dot{q}^\alpha}\right|_{q=\tilde{q}(t)}\right) - \left.\frac{\partial \omega}{\partial q^\alpha}\right|_{q=\tilde{q}(t)} - \left(\tilde{\omega} \wedge \left.\frac{\partial \omega}{\partial \dot{q}^\alpha}\right|_{q=\tilde{q}(t)}\right)\right] \wedge (\tilde{\mathbf{x}}-\tilde{\mathbf{c}}) = \mathbf{0}. \tag{2.33}$$

Since the point o can be chosen arbitrarily, the vector $(\tilde{\mathbf{x}}-\tilde{\mathbf{c}})$ can be assigned any possible value. In virtue of this we infer (2.29).

3. Virtual displacements

The Lagrangian formalism is closely related to the concept of virtual displacements. We introduce these in the following way. Considering the system at a certain time t, we assume the momentary values of the generalized coordinates and velocities to be known : $q^\alpha = \tilde{q}^\alpha(t)$ and

$\dot{q}^\alpha = \dot{\bar{q}}^\alpha(t)$, $\alpha = 1, 2, ..., n$. Then we determine the kinematical effects occurring if one of the elements of $\mathbf{q}$, say q^β, is changed by an infinitesimal amount δq^β, while keeping the other elements of $\mathbf{q}$, $\dot{\mathbf{q}}$ and t constant. We call δq^β a virtual change and from (2.13) and (2.16) we calculate the increments

$$\delta \mathbf{c} = \frac{\partial \mathbf{c}}{\partial q^\beta}\, \delta q^\beta, \quad \delta Q = \frac{\partial Q}{\partial q^\beta}\, \delta q^\beta, \qquad (\beta \text{ not summed}), \tag{3.1}$$

and the virtual displacement $\delta \mathbf{x}$ of a point P of the body $\mathscr{B}$

$$\delta \mathbf{x} = \frac{\partial \mathbf{x}}{\partial q^\beta}\, \delta q^\beta = \frac{\partial \mathbf{c}}{\partial q^\beta}\, \delta q^\beta + \frac{\partial Q}{\partial q^\beta}\, \xi \delta q^\beta, \qquad (\beta \text{ not summed}). \tag{3.2}$$

Similar to (2.8) and (2.10) we define the skew-symmetric tensor $\delta \Phi$

$$\delta \Phi = (\delta Q) Q^T, \tag{3.3}$$

and introduce its axial vector $\delta \varphi$

$$\delta \varphi_i = -\tfrac{1}{2} e_{ijk}\, \delta \Phi_{jk} = -\tfrac{1}{2} e_{ijk} \frac{\partial Q_{jl}}{\partial q^\beta} Q_{kl} \delta q^\beta, \quad (\beta \text{ not summed}). \tag{3.4}$$

Then we can write (3.2) in the form

$$\delta \mathbf{x} = \delta \mathbf{c} + \delta \varphi \wedge (\mathbf{x} - \mathbf{c}), \tag{3.5}$$

in which $\delta \mathbf{c}$ is the virtual translation and $\delta \varphi$ is the virtual rotation pertaining to δq^β. In the sequel we shall apply the relation

$$\delta \varphi = \frac{\partial \omega}{\partial \dot{q}^\beta}\, \delta q^\beta, \tag{3.6}$$

which follows from $(3.4)^2$ and $(2.26)^3$,

For the sake of completeness and for later reference an expression for the virtual work $\delta \mathscr{A}$ is included in this section. To this end we assume that the field of external forces per unit of volume $\mathbf{K}$ and moments per unit of volume $\mathbf{M}$ exerted on the body is known. Without loss of generality we choose the representation

$$\mathbf{x} \in V, \quad t \in \mathbb{R}, \quad \mathbf{K} = \mathbf{K}(\mathbf{x}, t) \quad \text{and} \quad \mathbf{M} = \mathbf{M}(\mathbf{x}, t), \tag{3.7}$$

in which V is the region occupied by $\mathscr{B}$ at time t, and $\mathbf{x}$ is the point of application of $\mathbf{K}$ and $\mathbf{M}$, respectively. By definition the virtual work is

$$\delta \mathscr{A} = \int_V (\mathbf{K}, \delta \mathbf{x})\, dV + \int_V (\mathbf{M}, \delta \varphi)\, dV, \tag{3.8}$$

which with the use of (3.5) can be written as

$$\delta\mathscr{A} = \int_V \{(\mathbf{K}, \delta\mathbf{c}) + ((\mathbf{x}-\mathbf{c}) \wedge \mathbf{K} + \mathbf{M}, \delta\boldsymbol{\varphi})\}dV$$
$$= (\mathscr{K}, \delta\mathbf{c}) + (\mathscr{M}_0, \delta\boldsymbol{\varphi}), \tag{3.9}$$

where $\mathscr{K} = \int_V \mathbf{K}dV$ and $\mathscr{M}_0 = \int_V \{(\mathbf{x}-\mathbf{c}) \wedge \mathbf{K} + \mathbf{M}\}dV$

are the resultant force and the resultant moment with respect to o on the body, respectively. Using (3.1)[1] and (3.6) in (3.9) we find for the generalized force $\bar{Q}_\beta$ defined by

$$\delta\mathscr{A} = \bar{Q}_\beta \delta q^\beta, \quad (\beta \text{ not summed}), \tag{3.10}$$

the expression

$$\bar{Q}_\beta = \left(\mathscr{K}, \frac{\partial \mathbf{c}}{\partial q^\beta}\right) + \left(\mathscr{M}_0, \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}^\beta}\right). \tag{3.11}$$

4. Kinetic energy

From now on we choose the origin of the moving frame o in the centre of gravity of the body This means that for every t

$$\int_V (\mathbf{x}-\mathbf{c})\rho(\mathbf{x}, t)dV = \mathbf{0}, \tag{4.1}$$

in which $\rho(\mathbf{x}, t)$ is the density defined on $\mathbf{x}$ and t. By definition the kinetic energy T is

$$T = \tilde{T}(t) = \tfrac{1}{2} \int_V (\tilde{\mathbf{v}}(t), \tilde{\mathbf{v}}(t))\rho dV, \tag{4.2}$$

into which we can substitute (2.12). In view of the structure of the expression obtained in this way we define

$$T(\mathbf{q}, \dot{\mathbf{q}}, t) = \tfrac{1}{2} m(\dot{\mathbf{c}}, \dot{\mathbf{c}}) + \tfrac{1}{2}(\boldsymbol{\omega}, J\boldsymbol{\omega}), \tag{4.3}$$

where we have written for short

$$\dot{\mathbf{c}} = \frac{\partial \mathbf{c}}{\partial q^\beta}\dot{q}^\beta + \frac{\partial \mathbf{c}}{\partial t}, \quad m = \int_V \rho dV = \text{constant}$$

and

$$J = \int_V [(\mathbf{x}-\mathbf{c}, \mathbf{x}-\mathbf{c})I - (\mathbf{x}-\mathbf{c}) \otimes (\mathbf{x}-\mathbf{c})]\rho dV.$$

The scalar m is the mass of $\mathscr{B}$ and the linear form J is its central inertia tensor. We note the relation

$$T(\tilde{q}(t), \dot{\tilde{q}}(t), t) = \tilde{T}(t).$$

The first term in the right-hand side of (4.3)[1] is called the translational kinetic energy, and the second one the rotational energy. The components of J with respect to the fixed basis and the moving one are denoted by J_{ij} and J'_{ij}, respectively, yielding the representations

$$J = J_{ij}\,\mathrm{e}_i \otimes \mathrm{e}_j, \quad \text{with}$$
$$J_{ij} = \int_V [(x_k - c_k)(x_k - c_k)\delta_{ij} - (x_i - c_i)(x_j - c_j)]\,\rho dV, \tag{4.4}$$

and

$$J = J'_{ij}\,\mathrm{f}_i \otimes \mathrm{f}_j, \quad \text{with} \tag{4.5}$$
$$J'_{ij} = \int_V (\xi_k \xi_k \delta_{ij} - \xi_i \xi_j)\,\rho dV.$$

Obviously

$$J_{ij} = J_{ij}(\mathrm{q}, t), \tag{4.6}$$

while the components J'_{ij} are constants. Finally, we note the relation

$$J_{ij} = Q_{ik} Q_{jl} J'_{kl}, \tag{4.7}$$

which follows from (4.4)[2], (4.5)[2] and (2.1), or can be inferred from Cartesian tensor transformation rules.

5. Statement of the problem

For the moment being we consider a material system consisting of N_1 mass points m_1^i, $i = 1, 2, \ldots, N_1$, and N_2 rigid bodies (masses m_2^i, central inertia tensors J^i, $i = 1, 2, \ldots, N_2$). As before we suppose that we can calculate the position and the velocity of each material point, whether a mass point or a point of a body, from n generalized coordinates q^α, n generalized velocities $\dot{q}^\alpha$ and the time t. In the usual way we obtain an expression for the kinetic energy T in the form $T = T(\mathrm{q}, \dot{\mathrm{q}}, t)$. We observe that three types of terms contribute to it: the kinetic energy T_1 of the mass points, the total translational energy T_2 of the rigid bodies and, finally, their total rotational energy T_3

$$T = T_1 + T_2 + T_3. \tag{5.1}$$

Our aim is to derive Lagrange's equations

$$\frac{d}{dt}\left(\left.\frac{\partial T(\mathrm{q}, \dot{\mathrm{q}}, t)}{\partial \dot{q}^\beta}\right|_{\mathrm{q}=\tilde{\mathrm{q}}(t)}\right) - \left.\frac{\partial T(\mathrm{q}, \dot{\mathrm{q}}, t)}{\partial q^\beta}\right|_{\mathrm{q}=\tilde{\mathrm{q}}(t)} = \bar{Q}_\beta(\tilde{\mathrm{q}}(t), t), \quad (\beta = 1, 2, \ldots, n) \tag{5.2}$$

from the balance laws for mass points and rigid bodies. In order to simplify further analysis we note that, similar to (5.1), there are three contributions to the generalized force $\bar{Q}_\beta$, viz., $\bar{Q}_{\beta 1}$ resulting from the forces exerted on the mass points, $\bar{Q}_{\beta 2}$ contributed by the forces applied to the bodies and, finally, $\bar{Q}_{\beta 3}$ as a result of the moments (with respect to the distinct centres of gravity of the bodies) acting on the rigid bodies

$$\bar{Q}_\beta = \bar{Q}_{\beta 1} + \bar{Q}_{\beta 2} + \bar{Q}_{\beta 3}. \tag{5.3}$$

In view of this and (5.1) we can prove (5.2) by establishing its validity for the quantities T and $\bar{Q}_\beta$ labelled 1, 2 and 3 separately. Since the proofs for the labels 1 and 2 are trivial and can be found in a multitude of textbooks, we dispense with them here. (We note that the identities $(2.26)^1$ and $(2.26)^4$ are to be used only). Hence we can confine our attention to the proof of (5.2) for T_3 and $\bar{Q}_{\beta 3}$, pertaining to the rotations of the bodies about their centres of gravity. Observing that the distinct bodies contribute to T_3 and $Q_{\beta 3}$ additively as well, we single out one rigid body without loss of generality. In this way we return to the previous sections the results of which are to be applied to a rigid body that rotates about its centre of gravity, i.e.,

$$\mathbf{c} = \mathbf{0}. \tag{5.4}$$

We shall prove (5.2) starting from the balance law of angular momentum referred to the centre of gravity of the body

$$\frac{d}{dt}(\tilde{J}(t)\,\tilde{\omega}(t)) = \int_V \{\tilde{\mathbf{x}}(t) \wedge \tilde{\mathbf{K}}(t) + \tilde{\mathbf{M}}(t)\}\,dV = \mathscr{M}_0(t), \tag{5.5}$$

where the meaning of the various quantities and symbols follows from Sections 3 and 4.

Let us now consider a virtual rotation $\delta\varphi$ resulting from the virtual change δq^β. Taking the inner product of $\delta\varphi$ and both sides of (5.5), we find

$$\left(\frac{d}{dt}(\tilde{J}\,\tilde{\omega}), \delta\varphi\right) = \bar{Q}_\beta \delta q^\beta, \quad (\beta \text{ not summed}), \tag{5.6}$$

where we have used (5.4), (3.9) and (3.10). On comparing (5.6) with (5.2), after the elimination of $\delta\varphi$ from (5.6) with the aid of (3.6), we see that we have to prove

$$\frac{d}{dt}\left(\left.\frac{\partial T}{\partial \dot{q}^\alpha}\right|_{\mathrm{q}=\tilde{\mathrm{q}}(t)}\right) - \left.\frac{\partial T}{\partial q^\alpha}\right|_{\mathrm{q}=\tilde{\mathrm{q}}(t)} - \left(\frac{d}{dt}(\tilde{J}\,\tilde{\omega}), \left.\frac{\partial \omega}{\partial \dot{q}^\alpha}\right|_{\mathrm{q}=\tilde{\mathrm{q}}(t)}\right) = 0,$$
$$(\alpha = 1, 2, \ldots, n). \tag{5.7}$$

To this end we substitute $(4.3)^1$ and (5.4) in (5.7). In this way the latter is transformed into

$$\left(\tilde{J}\,\tilde{\omega}, \frac{d}{dt}\left(\left.\frac{\partial \omega}{\partial \dot{q}^\alpha}\right|_{\mathrm{q}=\tilde{\mathrm{q}}(t)}\right)\right) - \frac{1}{2}\left.\left(\frac{\partial}{\partial q^\alpha}(J\omega, \omega)\right)\right|_{\mathrm{q}=\tilde{\mathrm{q}}(t)} = 0, \quad (\alpha = 1, 2, \ldots, n). \tag{5.8}$$

In the next two sections we shall prove (5.8) in two ways.

6. Proof with the use of coordinates

In this section we start from the representation for J on the fixed basis according to (4.4). Using (4.6) we transform (5.8) into

$$\left(\tilde{J}\tilde{\omega}, \frac{d}{dt}\left(\frac{\partial\omega}{\partial\dot{q}^\alpha}\bigg|_{q=\tilde{q}(t)}\right) - \frac{\partial\omega}{\partial q^\alpha}\bigg|_{q=\tilde{q}(t)}\right) - \tfrac{1}{2}\left(\frac{\partial J}{\partial q^\alpha}\bigg|_{q=\tilde{q}(t)} \tilde{\omega}, \tilde{\omega}\right)$$

$$= \left(\tilde{J}\tilde{\omega}, \frac{d}{dt}\left(\frac{\partial\omega}{\partial\dot{q}^\alpha}\bigg|_{q=\tilde{q}(t)}\right) - \frac{\partial\omega}{\partial q^\alpha}\bigg|_{q=\tilde{q}(t)}\right) - \left(\tilde{J}\tilde{\omega}, \tilde{Q}\,\frac{\partial Q^T}{\partial q^\alpha}\bigg|_{q=\tilde{q}(t)} \tilde{\omega}\right) = 0. \quad (6.1)$$

With the use of (2.24) this can be worked out further into

$$(J\omega)_i \left[-e_{ijk}\frac{\partial Q_{jp}}{\partial q^\alpha}\dot{Q}_{kp} + \tfrac{1}{2}e_{mkl}Q_{ij}\frac{\partial Q_{mj}}{\partial q^\alpha}Q_{lp}\dot{Q}_{mp}\right] = 0, \quad (6.2)$$

(where we have omitted the explicit reference to the actual motion $q = \tilde{q}(t)$). Since this expression has to vanish for all values of the components of $J\omega$, the form between square brackets must be equal to zero identically

$$-e_{ijk}\frac{\partial Q_{jp}}{\partial q^\beta}\dot{Q}_{kp} + \tfrac{1}{2}e_{mkl}Q_{ij}\frac{\partial Q_{mj}}{\partial q^\beta}Q_{lp}\dot{Q}_{kp} = 0. \quad (6.3)$$

To prove this identity we rewrite the first term on the left-hand side through the use of (2.11) and (2.18) as follows

$$-e_{ijk}\frac{\partial Q_{jp}}{\partial q^\beta}\dot{Q}_{kp} = \tfrac{1}{2}e_{ijk}Q_{lp}Q_{jr}\frac{\partial Q_{lr}}{\partial q^\beta}\dot{Q}_{kp} + \tfrac{1}{2}e_{ijk}\frac{\partial Q_{jp}}{\partial q^\beta}Q_{kr}Q_{lp}\dot{Q}_{lr}.$$

Then we interchange the indices r and p in the last term and use (2.17), obtaining

$$-e_{ijk}\frac{\partial Q_{jp}}{\partial q^\beta}\dot{Q}_{kp} = -\tfrac{1}{2}e_{ijk}Q_{jr}\frac{\partial Q_{lr}}{\partial q^\beta}(Q_{kp}\dot{Q}_{lp} - Q_{lp}\dot{Q}_{kp})$$

$$= -\tfrac{1}{2}e_{ijk}Q_{jr}\frac{\partial Q_{lr}}{\partial q^\beta}e_{klm}e_{mts}Q_{tp}\dot{Q}_{sp}$$

$$= \tfrac{1}{2}\left(e_{jts}Q_{jr}\frac{\partial Q_{ir}}{\partial q^\beta} - e_{its}Q_{jr}\frac{\partial Q_{jr}}{\partial q^\beta}\right)Q_{tp}\dot{Q}_{sp}.$$

We note that the second term is zero. Using (2.17)[1] again and relabeling some dummy indices, we retain the first term in the form

$$-\tfrac{1}{2}e_{mkl}Q_{ij}\,\frac{\partial Q_{mj}}{\partial q^{\beta}}\,Q_{lp}\dot{Q}_{kp}. \tag{6.4}$$

With (6.4) the left-hand side of (6.3) vanishes identically. Hence, we have proved (5.7).

Alternatively, we can proceed in a less formal way to verify this identity. To this end we define the skew-symmetric tensor A by

$$A = \frac{\partial Q}{\partial q^{\beta}}\,Q^{T}. \tag{6.5}$$

With (6.5) and (2.8), the definition of Ω, the left-hand side of (6.3) can be written in the form

$$-e_{ijk}A_{jp}\Omega_{kp} + \tfrac{1}{2}A_{mi}e_{mkl}\Omega_{kl}. \tag{6.6}$$

Elimination of Ω from this expression with the aid of (2.10)[1] yields

$$e_{ijk}A_{jp}e_{kpl}\omega_{l} - A_{mi}\omega_{m} = A_{ji}\omega_{j} - A_{jj}\omega_{i} - A_{ji}\omega_{j}.$$

Since A is skew-symmetric this expression is equal to zero identically. Hence, we have proved (6.3) again.

7. Proof without using coordinates

We start from (5.8) in the form

$$\left(J\omega, \frac{d}{dt}\left(\frac{\partial\omega}{\partial\dot{q}^{\beta}}\right)\right) - \frac{\partial}{\partial q^{\beta}}\,(\tfrac{1}{2}J\omega, \omega) = 0, \tag{7.1}$$

where by definition, and since $\mathbf{c} = 0$,

$$J\omega = \int_V \rho\,\mathbf{x}\wedge(\omega\wedge\mathbf{x})\,dV. \tag{7.2}$$

(Again, we delete the independent variables q^{α}, $\dot{q}^{\alpha}$ and t, and do not refer to the motion $\mathbf{q}=\tilde{\mathbf{q}}(t)$ explicitly). From (7.2) we find

$$\frac{\partial}{\partial q^{\beta}}\,(\tfrac{1}{2}J\omega, \omega) = \left(J\omega, \frac{\partial\omega}{\partial q^{\beta}}\right) + \left(\omega, \int_V \rho\left(\frac{\partial\mathbf{x}}{\partial q^{\beta}}\wedge(\omega\wedge\mathbf{x})\right)dV\right).$$

Using (2.28) with (5.4) together with the vectorial identity

$$\forall_{\mathbf{e}}\ (\omega, \{(\mathbf{e}\wedge\mathbf{x})\wedge(\omega\wedge\mathbf{x})\}) - (\mathbf{x}, \{(\mathbf{e}\wedge\omega)\wedge(\omega\wedge\mathbf{x})\}) = 0$$

this relation is readily transformed into

$$\frac{\partial}{\partial q^\beta}\left(\tfrac{1}{2}(J\omega,\omega)\right)=\left(J\omega,\frac{\partial\omega}{\partial q^\beta}\right)+\int_V\left(\mathrm{x},\left\{\left(\frac{\partial\omega}{\partial\dot{q}^\beta}\wedge\omega\right)\wedge(\omega\wedge\mathrm{x})\right\}\right)\rho dV$$

$$=\left(J\omega,\frac{\partial\omega}{\partial q^\beta}\right)+\int_V\rho\left(\left(\frac{\partial\omega}{\partial\dot{q}^\beta}\wedge\omega\right),\quad(\mathrm{x}\wedge(\mathrm{x}\wedge\omega))\right)dV. \tag{7.3}$$

On using the definition (7.2) of J we rewrite (7.3) as

$$\frac{\partial}{\partial q^\beta}\left(\tfrac{1}{2}(J\omega,\omega)\right)=\left(J\omega,\frac{\partial\omega}{\partial q^\beta}\right)-\left(\left(\frac{\partial\omega}{\partial\dot{q}^\beta}\wedge\omega\right),J\omega\right).$$

Substituting this in (7.1) and applying (2.29) yields

$$\left.\left(J\omega,\left\{\frac{d}{dt}\left(\frac{\partial\omega}{\partial\dot{q}^\beta}\right)-\frac{\partial\omega}{\partial q^\beta}-\left(\omega\wedge\frac{\partial\omega}{\partial\dot{q}^\beta}\right)\right\}\right)\right|_{\mathrm{q}=\tilde{\mathrm{q}}(t)}\equiv 0.$$

Evidently (7.1) vanishes and this proves that (5.7) is true.

8. Discussion

We have proved the validity of (5.7) in two ways in Sections 6 and 7, respectively. As has been sketched in Section 5, we can infer Lagrange's equations (5.2) from it. On comparing the methods used in Sections 6 and 7, we see that the analysis without the use of coordinates is very simple and is to be preferred for that reason above the application of coordinates for the rotational motion, whether Eulerian angles [3] or components of an orthogonal tensor (Section 6) are used.

REFERENCES

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